AN APPROACH TO THE SYNTHESIS OF LINEAR NETWORKS THROUGH USE OF NORMAL COORDINATE TRANSFORMATIONS LEADING TO MORE GENERAL TOPOLOGICAL CONFIGURATIONS*

Ernst A. Guillemin
Massachusetts Institute of Technology
Cambridge 39, Massachusetts

SUMMARY

The proposed procedure presents a method of determining parameter matrices from given impedance functions through use of normal coordinate transformations, and realizes the pertinent network by identical tree configurations in single-element-kind networks having a general topological structure. Essentially the same procedure is applicable to passive bilateral networks and to non-passive and/or non-bilateral ones, differing only in the appropriate normal coordinate transformation. Although tedious computations are involved, the availability of modern computers makes this method feasible and thus opens up a more general and potentially useful approach to network synthesis.

INTRODUCTION

In the realization procedures that have so far been devised, parameter matrices and pertinent general topological network configurations play no part. Instead the function to be realized is developed in some fashion to which a physical configuration of branches may be associated by inspection. The chief weakness of these methods lies in the fact that the scope of topological configurations obtainable is extremely limited, with the result that our knowledge about economical use of elements or possible practical advantages of more general network configurations is correspondingly limited.

The proposed procedure presents a method of determining parameter matrices from the given impedance function (or functions) through use of normal coordinate transformations in a way that makes available a large degree of flexibility; and then realizes the pertinent network through first finding single-element-kind realizations for the separate parameter matrices with a completely general topological structure but with identical tree configurations for the several matrices. Connecting corresponding nodes in these trees then yields the total realization at once.

Essentially the same procedure is applicable to passive-bilateral networks and to non-passive and/or non-bilateral ones, the difference being only in the detailed nature of the appropriate normal coordinate transformations.

While interest at the moment may be focused upon active non-bilateral network synthesis, the new method of approach can also throw much needed light upon unsolved questions in passive bilateral synthesis, such as whether a general RLC d.p. impedance realization must necessarily involve large element numbers as does the Bott and Duffin method, or the question as to what are the limitations of a grounded two terminal-pair (three terminal network) for the realization of transfer impedances of RC or RLC networks, etc.

Determination of parameter matrices from given response functions.

In order not to confuse presentation of the basic idea of our procedure by including things that are momentarily irrelevant even though they are ultimately important, we shall consider at this point a situation that is both simple and well understood according to conventional methods of analysis and synthesis. As a tentative vehicle for presenting the proposed method, we therefore choose the passive bilateral RC network.

Realization that the same ideas can be applied to any linear system whether active or passive, bilateral or nonbilateral, characterized by three parameter matrices rather than by only two, will be aided by subsequent discussion of detailed considerations essential to extending the same procedure to more general systems.

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* This work was supported in part by the U. S. Army (Signal Corps), the U. S. Air Force (Office of Scientific Research, Air Research and Development Command), and the U. S. Navy (Office of Naval Research).

In our proposed approach the idea is to accomplish realization of the ultimate network from parameter matrices rather than from a given rational function or set of functions as we do in all of the presently known synthesis procedures. In contrast with the latter, we recognize at the outset that a method in which the construction of pertinent parameter matrices plays an essential part, can deal more effectively with the problem of network equivalence since that problem involves also the manipulation of parameter matrices and hence may be incorporated as an integral part of the realization procedure rather than have it appear as an appendage thereto.

Achievement of the proposed method depends upon the successful crossing of two major hurdles: (a) the construction of parameter matrices appropriate to a specified response function (or functions) and (b) a realization procedure that yields a network corresponding to given parameter matrices with designation of pertinent terminal pairs. Techniques for accomplishing these objectives are presented in the following discussion.

Consider an RC network characterized on the node basis in the usual manner in terms of a capacitance matrix C and a conductance matrix G, both of order n (the number of tree branches in a pertinent graph). If s denotes the complex frequency variable, then the so-called node admittance matrix is given by

$$Y = Cs + G, \qquad (1)$$

and the equilibrium equations read

$$Y \times E = I_s \qquad (2)$$

in which E is a column matrix involving the n node-pair voltage variables and I_s is a column matrix with elements equal to the current sources bridged across node pairs (tree branches).

By means of a linear transformation of variables we wish to introduce the normal coordinates of this system, which means, according to familiar methods,* that we wish through a congruent transformation to diagonalize simultaneously the matrices C and G. We can think of accomplishing this end in two separate steps. In the first we transform C (which we assume to be nonsingular) to its canonic form having all plus-one's on the principal diagonal and all other elements zero (the one's are all plus because the quadratic form associated with C is positive definite on account of the passive character of the network). This transformation affects G also but leaves it symmetrical and nondiagonal.

We next transform this resulting G matrix congruently with an orthogonal matrix so as to obtain its diagonalization, recognizing that the effect upon the canonic form of the C matrix is nil. If we now combine the two transformations into a single one with the matrix A, then we can summarize the total transformation by writing

$$A_t \times Y \times A = Y_d = \begin{bmatrix} (g_1+s) & 0 & \dots & 0 \\ 0 & (g_2+s) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (g_n+s) \end{bmatrix} \qquad (3)$$

where the diagonal admittance matrix Y_d characterizes the network in its normal coordinates, and g_1, g_2, ... g_n are the negatives of the natural frequencies of the network. Since they are the latent roots of a real symmetrical matrix their required positive realness is assured.

The response of our RC network is characterized by the inverse of the matrix Y in Eq. 2. According to Eq. 3 this inverse is given by

$$Y^{-1} = Z = A \times Y_d^{-1} \times A_t. \qquad (4)$$

But

$$Y_d^{-1} = Z_d = \begin{bmatrix} \frac{1}{g_1+s} & 0 & \dots & 0 \\ 0 & \frac{1}{g_2+s} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{g_n+s} \end{bmatrix} \qquad (5)$$

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* See, for example, "The Mathematics of Circuit Analysis", John Wiley 1949, pp 111-118.

and if we write for the matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (6)$$

then a typical element in the open-circuit impedance matrix Z is seen to be given by

$$z_{ik} = \sum_{\nu=1}^{n} \frac{k_\nu}{g_\nu + s} \qquad (7)$$

with

$$k_\nu = a_{i\nu}\, a_{k\nu}. \qquad (8)$$

We recognize 7 to be the partial fraction expansion of the impedance function z_{ik} and 8 are the residues in its poles. For a driving-point function (i = k), the residues are thus seen to fix the elements in one row of the matrix A, eq. 6; while for a transfer impedance the residue values fix only the products of corresponding elements in two rows.

In a synthesis problem in which one or more z_{ik} functions are given, their partial fraction expansions are thus seen to fix a certain portion of the A matrix. In the synthesis of a driving-point impedance, for example, one row of A is fixed by the given impedance function; all other elements in this matrix are free to choose in whatever manner best suits the realization of the resulting parameter matrices R and C.

In this regard we have from Eqs. 1 and 3

$$C = A_t^{-1} \times A^{-1} \qquad (9)$$

and

$$G = A_t^{-1} \times \begin{bmatrix} g_1 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & g_n \end{bmatrix} \times A^{-1} \qquad (10)$$

in which $g_1 \ldots g_n$ are determined from the poles of the specified impedance function.

If we designate the latter as z_{11} and choose for A the triangular form

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad (11)$$

then the inverse of A has the same form, namely

$$A^{-1} = B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{22} & b_{23} & \cdots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_{nn} \end{bmatrix} \qquad (12)$$

and we have the following relations between the a's and b's

$$a_{11}\, b_{11} = 1 \qquad (13)$$

$$\left.\begin{aligned} a_{11}\, b_{12} + a_{12}\, b_{22} &= 0 \\ a_{22}\, b_{22} &= 1 \end{aligned}\right\} \qquad (14)$$

$$\left.\begin{aligned} a_{11}\, b_{13} + a_{12}\, b_{23} + a_{13}\, b_{33} &= 0 \\ a_{22}\, b_{23} + a_{23}\, b_{33} &= 0 \\ a_{33}\, b_{33} &= 1 \end{aligned}\right\} \qquad (15)$$

and so forth.

Residues of z_{11} determine elements in the first row of the matrix 11 (except for their algebraic signs which can be plus or minus since the residues fix only the squares of these elements). All other elements a_{ik}, and hence a corresponding number of b_{ik}, may be chosen solely with regard to the realizability of the matrices C and G as single-element-kind n + 1-node networks with identical tree configurations.

The appropriateness of this realization technique is discussed further below. For the moment it will suffice to say that the pertinent realizability conditions are simple enough to

enable one to establish a systematic procedure for the choice of b_{ik} values in a sequential manner consistent with the form of relations 13, 14, 15 ..., and thus to construct a pair of parameter matrices C and G fulfilling these realizability conditions on the one hand while satisfying the demands of the stated impedance z_{11} on the other.

As might be expected, this process is not unique but offers a number of free choices to be made. Thus it may ultimately arrive at a variety of networks realizing the stated impedance function, and the network equivalence problem, therefore, becomes an integral part of the procedure. Its analytic form is essentially that of the linear transformation method first proposed by Cauer in 1928. Unlike the limited extent to which presently known synthesis procedures facilitate construction of equivalent networks by any method, and especially by the linear transformation of parameter matrices, the full potential of that method can now be achieved because the associated realization techniques are similarly geared to representations in terms of parameter matrices and the consideration of general topological configurations.

Active and/or non-bilateral elements

In the discussions of the preceding article, let us now assume that the RC network has embedded in it one or more resistive elements that are active and/or non-bilateral. This circumstance alters things only in that the matrix G is no longer symmetrical and perhaps that its associated quadratic form is not necessarily positive definite.* The procedure in this case involves only a minor variation from that just given.

We begin again with the relations expressed by 1 and 2, and subject Y to a congruent transformation (like that given in the left-hand side of Eq. 3) in which the matrix A now, however, only yields C in canonic form and leaves G a general non-symmetric matrix, say $\bar{G}$. If g_1, g_2, ... g_n are the latent roots of $\bar{G}$ and B denotes the pertinent modal matrix** fulfilling the relation

$$B^{-1} \times \bar{G} \times B = \begin{bmatrix} g_1 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & g_n \end{bmatrix} \qquad (16)$$

then the diagonalization of Y analogous to that in Eq. 3 is now expressed by

$$P_t \times Y \times Q = Y_d = \begin{bmatrix} (g_1+s) & 0 & \cdots & 0 \\ 0 & (g_2+s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (g_n+s) \end{bmatrix} \qquad (17)$$

where

$$P = A \times B_t^{-1} \text{ and } Q = A \times B \qquad (18)$$

Eq. 4 is now replaced by

$$Y^{-1} = Z = Q \times Y_d^{-1} \times P_t, \qquad (19)$$

Eqs. 5 and 7 hold as before, and instead of 8 we now have the residues in 7 given by

$$k_\nu = q_{i\nu} p_{k\nu} \qquad (20)$$

in which $q_{i\nu}$ and $p_{k\nu}$ denote elements of Q and P respectively.

Since the latent roots of a real, non-symmetrical matrix are in general complex, we see that the z_{ik} in 7 now may have complex poles; and we also see that this result is due to the non-bilaterality of our RC network since it is the reason for the non-symmetry of G or $\bar{G}$. The present analysis is thus seen to agree in this respect with other methods for dealing with the same problem.

Regarding parameter matrices, Eq. 9 still holds, and with 18 gives

$$C = A_t^{-1} \times A^{-1} - P_t^{-1} \times Q^{-1} \qquad (21)$$

while 17 shows that now

$$G = P_t^{-1} \times \begin{bmatrix} g_1 & 0 & 0 & \cdots & 0 \\ 0 & g_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & g_n \end{bmatrix} \times Q^{-1} \qquad (22)$$

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* A special situation of this sort was considered in the paper entitled, "Transformation Theory Applied to Linear Active and/or Non-bilateral Networks", IRE PGCT Transactions Vol CT-4, Sept. 1957, pp. 106-111. The present discussion is in a sense a generalization of the basic ideas presented there.

** See "The Mathematics of Circuit Analysis", John Wiley, 1949, pp. 111-113.

Since this conductance matrix represents not only the passive resistive branches but also the embedded active non-bilateral devices (vacuum tubes or transistors), it should be regarded as a sum

$$G = G_1 + G_2 \qquad (23)$$

in which G_1 is symmetrical and realizable by an n + 1-node network with passive elements while G_2 has the structure and element values pertinent to one or more physical vacuum tubes or transistors connected to the same node pattern. This decomposition of G into additive parts, since it obviously is not unique, injects additional flexibility into the construction of matrices P and Q (or of A and B).

In this connection it should be observed that while the latent roots of a real dissymmetrical matrix like $\bar{G}$, Eq. 16, are in general complex, they need not be, and neither need the matrix B be complex. When it is complex, then P and Q in 18 are of course also complex; but one can approach the synthesis problem by constructing either real or complex matrices P and Q. In this regard we have a number of alternative possibilities to consider.

(a) We can construct real matrices P and Q, forming a dissymmetrical matrix $\bar{G}$ with real latent roots. However, any assumed driving-point or transfer impedance having the partial fraction expansion 7 with residues given by 20 may have zeros in the complex plane. Hence a voltage transfer ratio like $E_2/E_1 = z_{12}/z_{11}$ can have zeros and poles in the complex plane. The real latent roots $g_1 \ldots g_n$ of $\bar{G}$, which are poles of z_{11} and z_{12}, have no direct influence upon the transfer ratio E_2/E_1 and may be chosen in any way that facilitates realizability of the pertinent parameter matrices.

(b) We can assume complex poles for a desired transfer impedance z_{12} and prescribe these as the latent roots of $\bar{G}$. The construction of P and Q then involves complex elements since those of the matrix B are complex, although A remains real; and, of course, G, according to Eq. 22, must be real. Assuming a triangular form for P and Q and hence for P^{-1} and Q^{-1}, again facilitates the construction of these matrices although their complex character adds to the computational labor particularly where some cut-and-try is involved. Such difficulties may, however, be resolved with the help of appropriate computing facilities.

(c) In either of these attitudes we may allow the matrix G to have a general dissymmetrical form or, through suitable restriction upon the forms of A and B, construct a matrix G that is symmetrical except for an embedded 2 x 2 submatrix corresponding to a single vacuum tube or transistor. In other words, one can attempt to minimize the number of active and non-bilateral elements involved.*

Sufficiency of single-element-kind realization techniques.

Before pointing out how these ideas can be extended to the synthesis of linear networks containing all three kinds of elements, R L and C, plus vacuum tubes or transistors, it is appropriate to discuss the realization of such networks when their parameter matrices are known. These we assume to be of order n appropriate to equilibrium equations on the node basis for node-pair voltages identified with the branch voltages of a chosen tree in a network graph involving n + 1 nodes.

Let us consider for a moment that only a single parameter matrix, say G of order n, is specified. The simplest network for its realization has n + 1 nodes since the pertinent graph must at least have a tree with n branches. This same matrix G may, of course, be realized by a network involving more than n + 1 nodes; in fact such a more complicated network may realize a G matrix not realizable with an n + 1-node graph. As we shall point out presently, however, this more general realization is not acceptable for our purposes, and so only the n + 1-node realization techniques are pertinent to the problem considered here.

Realization of a single parameter matrix (for example, a G matrix) by an n + 1-node network has recently been accomplished,** and the conditions for its realizability are

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* This attitude was taken in the PGCT paper referred to above.

** See for example, "Applications of Matrix Algebra to Network Theory" by I. Cederbaum IRE Trans. on Circuit Theory, Vol. CT-6 May 1959, p. 127. A simpler procedure more adaptable to the needs of the present problem is given in the MIT Res. Lab. of Electronics Q.P.R. for Jan. 15, 1960 entitled, "On the Synthesis of Single-Element-Kind Networks" by E. A. Guillemin.

expressible in a variety of ways, some of them being adaptable to the needs of the methods of matrix construction discussed above. We shall not include the details of these realization procedures here but confine our discussion to showing that the n + 1-node single-element-kind realization process is all that we need for the construction of three-element-kind networks from their parameter matrices pertinent to a node basis.

When a single matrix like G is given and a network realization involving only positive elements and no ideal transformers is asked for then a necessary condition for the existence of this network is that G be appropriate to some geometrical tree configuration. Let us clarify this statement by reminding the reader that in an analysis problem where the network is given and we are asked to construct G corresponding to a set of equilibrium equations, it is first necessary to choose voltage variables. These may be the branch voltages of a tree, or they may be any linear independent combinations of branch voltages or node-pair voltages. In the latter instance, the resulting G matrix is not based upon a tree; in the former one it is. A necessary condition that a given G matrix be n + 1-node realizable without ideal transformers is that it be based upon a tree.*

In the synthesis of RC networks discussed above, both matrices G and C must be based upon the same geometrical tree configuration (a condition that is easy to meet in the process of matrix construction). In the synthesis of RLC networks where three parameter matrices are involved, all three must separately be n + 1-node realizable with the same geometrical tree configuration. Realization of the total network is accomplished in two steps. First each matrix is realized by a single-element-kind network for the implied common tree configuration, and then the several networks are joined by connecting corresponding tree branches in parallel (placing the trees in parallel by soldering together their corresponding nodes).

The fact that this procedure yields a network with the desired admittance matrix Y is easy to see. Consider, for example, the RC network situation represented by Eq. 1. This is an n terminal-pair network whose Y matrix is given by the sum of two component Y matrices, namely the matrices Cs and G. If two separate n terminal-pair networks having these component Y matrices are found, then their parallel combination has a resultant Y matrix equal to the sum of the component matrices provided certain conditions assuring no circulating currents resulting from the parallel connection** are fulfilled. It is readily established that these conditions are automatically fulfilled if the n + 1-node realizations for the separate matrices C and G are based upon the same geometrical tree configuration.

The schematic diagram in Fig. 1 illustrates the principle involved here. Straightforward extension of the well-known tests** for two terminal-pair networks, amounts to applying a voltage source E_s to one of the parallelled pairs of terminals as shown in the figure, separately shorting all other terminal pairs, and then testing for zero voltage between those shorted pairs like a-a', b-b', that are to be connected in parallel. This test must be repeated n times with the source E_s across each terminal pair. If the terminal pairs are across corresponding branches of identical tree configurations then their mutual adjacency (which insures a common terminal between pairs) is readily seen to assure that all tests yield zero voltage as required.

Therefore, when we approach the synthesis problem from a parameter matrix point of view rather than a conventional one in which the procedure yields a network configuration directly from the given impedance function, all we need for realization of the resulting network is a technique for constructing a single-element kind network from a given parameter matrix. It is rather surprising to find that the latter problem, which lay neglected for so many years (perhaps because no one considered it of sufficient practical interest to bother with), should become the tool for synthesis of completely general networks with unrestricted topological configuration!

Extension to general passive or active systems.

As is shown in art. 1 above, the key to the construction of parameter matrices from given rational impedance or response functions lies in the normal coordinate transformation of equilibrium equations, yielding the diagonalization of the pertinent admittance matrix as expressed by Eq. 3, for upon this artifice is based the ability to relate residues in the partial fraction expansion 7 with elements in

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* There are simple ways of testing a given G matrix to see if it is based upon a tree. See the M.I.T. RLE QPR reference cited on previous page.

** See "Synthesis of Passive Networks" by E. A. Guillemin, John Wiley, pp 191-194.

the matrix A and hence with those in the pertinent parameter matrices. For this reason the possibility of applying this synthesis approach to more general linear systems depends upon our ability to carry through a normal coordinate transformation (or at least a simultaneous diagonalization of all pertinent parameter matrices) for such systems.

Since equilibrium equations for RLC networks normally involve three quadratic energy functions and these are not simultaneously reducible to sums of squares by any known mathematical process, the logical procedure is to put the equilibrium equations of a general linear network into a form in which only two parameter matrices are involved. A detailed account of how a normal coordinate transformation is thus accomplished for a general linear network is recorded elsewhere.* We shall content ourselves here with accepting the results of that discussion and applying the pertinent ones to our immediate problem.

Essentially the scheme whereby only two parameter matrices are involved in the equilibrium equations of a general lossy network lies in expressing these in terms of both loop currents and node-pair voltages rather than in terms of either one of these sets of variables alone as is customarily done. The network branches are assumed to be in the form of series RL and parallel GC combinations with arbitrary R's and G's so that resistances alone can be regarded as degenate forms of either type of branch. While suited for the accommodation of a common practical situation, this approach retains complete generality.

The resulting equilibrium equations are thus found to have the matrix form

$$\left\{\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} p + \begin{bmatrix} R & \gamma_{\ell n} \\ -\gamma_{n\ell} & G \end{bmatrix}\right\} \times \begin{bmatrix} i \\ e \end{bmatrix} = \begin{bmatrix} e_s \\ i_s \end{bmatrix} \qquad (24)$$

in which L, C, R, G are submatrices, $p = d/dt$, i and e are column matrices involving ℓ loop-current and n node-pair voltage variables, and e_s and i_s are column matrices representing voltage sources acting around loops and current sources bridged across node pairs. The submatrix $\gamma_{\ell n}$ having ℓ rows and n columns and its transpose $\gamma_{n\ell}$ depend only upon network topology (not upon any element values).

If vacuum tubes or transistors are embedded in the network their presence can be reflected in the nonsymmetric or nonpositive character of the submatrix G. The matrix with L and C is still symmetrical and pertinent to a positive definite quadratic form. Assume further that any linear constraint relations among the variables have been used to eliminate dynamically superfluous ones so that this quadratic form is nonsingular. Then we can straightforwardly construct a matrix P which congruently transforms the first matrix in 24 to its canonic form and the second to some other dissymmetrical matrix B, thus

$$P_t \times \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix} \times P = u_b = \text{unit matrix of order b} \qquad (25)$$

and

$$P_t \times \begin{bmatrix} R & \gamma_{\ell n} \\ -\gamma_{n\ell} & G \end{bmatrix} \times P = B = \text{dissymmetrical matrix of order and rank b} \qquad (26)$$

A co-linear transformation of B with its modular matrix Q now yields a diagonal matrix D (whose elements are the latent roots of B) and leaves the canonic matrix 25 unchanged. This co-linear transformation of B reads

$$Q^{-1} \times B \times Q = D = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & d_b \end{bmatrix} \qquad (27)$$

in which the diagonal elements, as in Eq. 16, are the complex natural frequencies of the pertinent network, and Q is in general complex also.

For the familiar substitutions

$$i = I e^{st}, \quad e = E e^{st}, \quad i_s = I_s e^{st}, \quad e_s = E_s e^{st} \qquad (28)$$

* See "The Normal Coordinate Transformation of a Linear System with an Arbitrary Loss Function" M.I.T., RLE QPR, January 15, 1960. Also submitted for publication in the Journal of Mathematics and Physics.

and the notation

$$U = \begin{bmatrix} E_s \\ \cdots \\ I_s \end{bmatrix} \quad ; \quad V = \begin{bmatrix} I \\ \cdots \\ E \end{bmatrix}, \tag{29}$$

$$W = \left[\begin{array}{c|c} L\,s + R & \gamma_{\ell n} \\ \hline -\gamma_{n\ell} & C\,s + G \end{array}\right], \tag{30}$$

the equilibrium Eqs. 24 are given by

$$W \times V = U \tag{31}$$

The transformations 25, 26, 27 yield

$$Q^{-1} \times P_t \times W \times P \times Q = W_d =$$

$$\begin{bmatrix} (d_1+s) & 0 & 0 & \cdots & 0 \\ 0 & (d_2+s) & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & (d_b+s) \end{bmatrix} \tag{32}$$

and for the solution of Eq. 31 we are interested in

$$W^{-1} = X = P\,Q\,W_d^{-1}\,Q^{-1}\,P_t \tag{33}$$

With the matrix X partitioned as indicated in

$$X = \left[\begin{array}{c|c} X_{\ell\ell} & X_{\ell n} \\ \hline X_{n\ell} & X_{nn} \end{array}\right] \tag{34}$$

the solution to 31 may be written in the form

$$\begin{aligned} X_{\ell\ell}\,E_s + X_{\ell n}\,I_s &= I \\ X_{n\ell}\,E_s + X_{nn}\,I_s &= E \end{aligned} \tag{35}$$

Observe that the elements in the submatrix $X_{\ell\ell}$ are short-circuit driving-point and transfer admittances; those in X_{nn} are open-circuit driving-point and transfer impedances; and those in $X_{\ell n}$ and $X_{n\ell}$ are dimensionless transfer functions. X may, therefore, appropriately be referred to as an immittance matrix.

Elements of the matrix X, like those of the impedance matrix 4 or 19, according to relations 32 and 33, have the same representation as the z_{ik}'s in 7, namely

$$x_{ik} = \sum_{\nu=1}^{b} \frac{k_\nu}{d_\nu + s} \tag{36}$$

with residues given by

$$k_\nu = h_{i\nu}\ t_{k\nu} \tag{37}$$

where $h_{i\nu}$ and $t_{k\nu}$ are elements in the matrices

$$H = P \times Q \quad \text{and} \quad T = P \times Q_t^{-1} \tag{38}$$

In terms of these we can express the parameter matrices (similar to the relations 21 and 22) by the equations

$$\left[\begin{array}{c|c} L & 0 \\ \hline 0 & C \end{array}\right] = P_t^{-1} \times P^{-1} = T_t^{-1} \times H^{-1} \tag{39}$$

and

$$\left[\begin{array}{c|c} R & \gamma_{\ell n} \\ \hline -\gamma_{n\ell} & G \end{array}\right] = T_t^{-1} \times \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & d_b \end{bmatrix} \times H^{-1} \tag{40}$$

If H and T (and hence H^{-1} and T^{-1}) are assumed in triangular form, then their construction from residue requirements and realizability conditions on the parameter matrices is again possible in an analogous manner to that discussed above for the RC passive or active networks.

Here, as in the simpler cases, we assume a full graph; however, now each node is connected with every other node by an inductive branch as well as by a capacitive branch. This structure is the most general one possible; the requirements of particular problems may conceivably be met by simpler more specially chosen configurations. These as well as many other details still remain to be evaluated.

Concluding Remarks

The workability of this new approach to the synthesis problem is largely still in the initial study stages although sufficient progress in this direction has already been made to reveal its superior qualities regarding flexibility and generality. Initially, the method was applied to the well understood RC passive driving-point synthesis problem* in order to see how the familiar canonic Cauer and Fosterlike forms are obtained through the method of matrix construction suggested above. This study has shown not only that straight-forward procedures can be formulated for such matrix construction but also that equivalent networks are easily obtained through slight variations in these procedures and hence (as mentioned in the foregoing discussion) that the equivalent network problem is actually an integral part of this new approach rather than being separated from it and unrelated to it as is the case with all other synthesis methods.

The synthesis of an RC transfer impedance in the form of a grounded two terminal-pair by this method is in progress. It is hoped that here the unrestricted topological configuration inherent in the new approach will either reveal greater generalities than have been found so far by conventional methods or confirm that the latter are indeed as general as may be.

Other studies are focused upon the synthesis of active RC networks. An important existing procedure here is that of John Linvill who adapts his well known negative impedance converter to the construction of such networks for prescribed transfer functions with arbitrary (but stable) pole-zero distributions. While Linvill's approach yields very satisfactory results from a practical standpoint, the structural form of the network is rather special, and one would like to know for practical as well as for purely academic reasons what a completely general approach to the same problem might reveal.

Regarding passive RLC networks, the driving-point impedance problem when mutual inductive coupling is to be avoided, is so far solved in general terms only by the Bott and Duffin procedure which, for rational functions of higher degree, leads to excessively large numbers of elements. Although we suspect that such large element numbers are not actually a necessary condition for avoidance of mutual coupling, this circumstance has not yet been demonstrated. More light shed upon this problem would be most welcome, and the approach proposed here offers possibilities in this respect.

Other problems in passive RLC synthesis which can stand further illumination deal with transfer function realization, the emphasis here again being (for practical reasons) upon the grounded two terminal-pair and upon realizations in which incidental dissipation plays an essential role rather than being partially accommodated through some trick device or other.

Finally, this new approach is sufficiently general to include any active and/or non-bilateral network so long as it is linear. We can even include active and/or non-bilateral L's and C's (not yet invented), for the two matrices in Eq. 24 can simultaneously be diagonalized through pre-and post-multiplication by suitably chosen matrices P and Q respectively, even when neither matrix in 24 is symmetrical or associated with a positive definite quadratic form.

It is hoped that the present article will motivate others to take part in the development of detailed techniques for the implementation of this approach to the solution of problems mentioned here or to other related ones.

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* See the article by Richard Duda in the M.I.T. RLE Quarterly Progress Report for January 15, 1960 under the Network Synthesis Section.

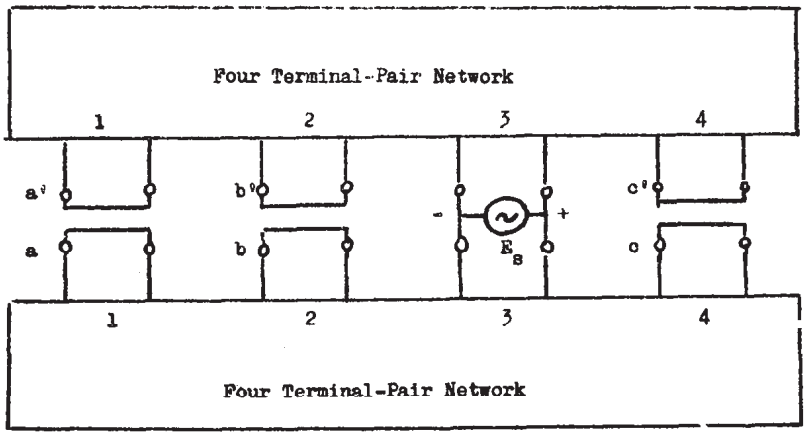

Fig. 1. Testing for fulfillment of conditions for which the resultant Y-matrix in a parallel connection of networks equals the sum of their individual Y matrices.